VOLUME 78 SEPARATE No. D-59

PROCEEDINGS

AMERICAN SOCIETY OF CIVIL ENGINEERS

FEBRUARY, 1952

AMERICAN SOCIETY OF CIVIL ENGINEERS FOUNDED 1852

DISCUSSION OF LIMIT DESIGN OF BEAMS AND FRAMES

(Published in February, 1951)

By L. E. Grinter, I. K. Silverman, Jack R. Benjamin, E. P. Popov, A. Hrennikoff, P. S. Symonds, H. Tachu, Vincenzo Franciosi, and H. J. Greenberg and W. Prager

STRUCTURAL DIVISION

Headquarters of the Society
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DISCUSSION

L. E. Grinter,[5] M. ASCE.—Using mathematical language this paper states two physical principles which engineers may prefer to consider without need of such rigorous proofs. The principles are readily established without mathematical equations. Their usefulness, as will be indicated, is limited by the need for considerable judgment in their application.

Consider any continous frame that supports a fixed load P. The frame is redundant to the $(n\text{-}1)$th degree. When n plastic hinges develop, collapse will begin and progress if a flat stress-strain curve is assumed. Choose by guess or by experience n locations for the successive development of n plastic hinges. This guess may be good or bad without influence on the conclusions. By simple geometry compute the angular discontinuities at the remaining n-1 plastic hinges when a unit angle change develops at one convenient hinge; also by geometry compute the corresponding movement of the load P in the direction of P. Then, for assumed collapse, the virtual work of the load, P(collapse) will equal the virtual work of M(yield) at the n hinges; and M(yield) may be constant or may vary from hinge to hinge, but it must be known. This equation of virtual work may be solved for P(collapse), the theoretical collapse load, if the n hinges are correctly located. However, since the hinge locations were merely assumed, the following applies:

> *Principle* (*a*).—If even one hinge is misplaced, that should have been at another point H, to preclude collapse it must be assumed that the steel near H had been locally hardened to increase M(yield). Therefore, the value of P(collapse), as computed for assumed hinge locations, is the collapse load for a frame locally hardened or strengthened at one or more points and is larger than P (collapse) for the frame with uniform properties. Therefore, the computed value of P(collapse) is an "upper bound."

The second principle is simplified even more than the first by adopting the physical viewpoint. Consider the frame as explained in (*a*) to be locally hardened $x\%$ at point H so that it does not collapse but is just at the point of collapse. The percentage x may be obtained from the moment diagram that is determined by statics with the aid of known yield moments at the n hinge points. Now reduce the yield point proportionately everywhere and reduce the value of P(collapse) computed in item (*a*) by the same proportion that will depend on the value of $x\%$ as indicated below. The frame still is on the point of collapse, but it has become a frame that has the original yield point only at point H and has a reduced yield point everywhere else. Hence the reduced value of the collapse load that is the original P(collapse) times $\left(\frac{100}{100 + x\%}\right)$ is an obvious lower bound on the collapse load. This principle may be stated as follows:

Note.—This paper by H. J. Greenberg and W. Prager was published in February, 1951, as *Proceedings-Separate No. 59*. The numbering of footnotes, equations, tables, and illustrations in this Separate is a continuation of the consecutive numbering used in the original paper.

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Principle (*b*).—If the frame from item (*a*), which has been locally hardened $x\%$ at point H and by smaller percentages at one or two other points, is then reduced everywhere in yield point by the ratio $\frac{100}{100 + x\%}$, it will return to its normal yield point only at point H and will be of reduced yield point everywhere else. This frame, which is much weaker than the original frame of uniform properties, is on the point of collapse under a reduced load $P(\text{collapse}) \frac{100}{100 + x\%}$. Hence this reduced load is a lower bound for the collapse load of the frame with uniform properties.

Limitations of Usefulness.—Although a single load P has been considered the application of the principle is equally evident for any fixed loading. The factor x is the greatest localized percentage increment in yield point needed to prevent collapse in item (*a*). No assumptions have been made as to the form or number of redundants in the frame. The main limitation upon usefulness is the need for very good engineering judgment. Unless the operator can almost guess the exact locations of the plastic hinges, the upper and lower bounds obtained are too widely separated to be very useful. For instance, it is not very helpful to know that a frame will probably collapse at 200,000 lb of loading but will not collapse at 100,000 lb of loading. An experienced engineer could study a set of plans and guess the collapse loading of a frame more closely than this without making such calculations.

The weakness of the principle seems to lie primarily in item (*b*). In item (*a*) the theoretical collapse load is completed for a frame that is of normal yield point everywhere, except at one or two local areas where it must be hardened, say x and $y\%$. In item (*b*) however, all of the frame is softened except near one plastic hinge below the original yield level. This reduction is determined from the larger percentage of local hardening, that is x or $y\%$. Hence the lower bound upon the collapse loading is an extremely conservative value which exaggerates the spread between upper and lower bounds as computed by examples (*a*) and (*b*). Of course, improved guesses can be used to reach better and better agreement between upper and lower bounds until they become identical and a solution is obtained. This, however, may be a slow process for complex structures, where such a technique would be most helpful.

These comments have served to show that the two principles established mathematically by the authors may be proved quite simply by physical reasoning. However, this approach does not change the fact that their convenient use in structural design depends upon the judgment of the engineer in guessing the approximate locations of the plastic hinges.

I. K. Silverman,[6] Assoc. M. ASCE.—A method for solving the problem of determining the load required to cause collapse in a structure of known dimensions is indicated in this paper. The reverse of this problem is designing for the dimensions of a structure that will give the greatest economy for given loads.

Most engineers deal with the latter problem. The method of analysis given by the authors furnishes answers, using an iterative process, to both

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problems. The engineer assumes a design, determines the safety factor (as defined by the authors), examines this safety factor in the light of the desired factor, and, if necessary, modifies his design. It will be found that there are a great many designs (relative dimensions) that will satisfy any criterion of safety factor. Further analyses as to relative economy will fix the final design.

Except for local failure the number of yield hinges necessary to transform a structure into a mechanism is one greater than the structure's degree of statical indeterminateness. But there is no theoretical reason why a structure cannot be so proportioned that simultaneous failure occurs in all elements of the structure with the occurrence of more than the minimum number of required yield hinges. This brings to mind the "one-hoss shay" principle:[7]

> "Have you heard of the wonderful one-hoss shay
> That was built in such a logical way
> It ran for a hundred years to a day?
> And then of a sudden * * *
> It went to pieces all at once—
> All at once and nothing first,
> Just as bubbles do when they burst."

The theory of limit design contains in itself a direct method for the design of structures on the "one-hoss shay" principle. Consider the beam of Fig. 14

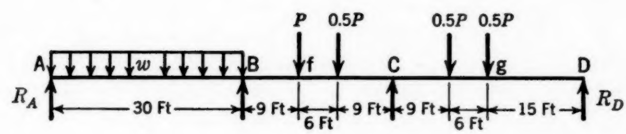

FIG. 14

carrying the fixed loads shown. The loads are expressed as a function of a single parameter P. The center span is designed to furnish three yield hinges located at B, C, and f. Assuming a constant limit moment at B, C, and f equal to M_2 the following equations may be written:

$$M_B + M_C = M_F = M_2 \qquad (15)$$

$$M_f + M_B - 9\,V_B = 0 \qquad (16)$$

$$M_B + 15\,P + 4.5\,P = M_c + 24\,V_B \qquad (17)$$

in which V_B is the shear to the right of the support B acting upwards. It follows that

$$V_B = \frac{2}{9}M_2 = \frac{19.5\,P}{24} \qquad (18a)$$

and

$$M_2 = 3.66\,P \qquad (18b)$$

The next step is to determine the value of the limit moment so that a hinge occurs at the point of the maximum bending, which in span AB is at the point

[7] "The Deacon's Masterpiece," by Oliver Wendell Holmes, *World's Best Loved Poems*, Harper and Bros., New York, N. Y., 1927, p. 362.

of zero shear. Denoting the maximum bending moment in span AB as M_x

$$M_x = R_A\, x - \frac{w\, x^2}{2} \quad \text{(19)}$$

$$X = R_A/w \quad \text{(20)}$$

$$M_B = 30\, R_A - 450\, w = -\, M_2 \quad \text{(21)}$$

These equations give

$$M_x = \frac{(15\, w - 0.122\, P)^2}{2\, w} \quad \text{(22)}$$

For span CD it is assumed that the maximum moment occurs under the load nearest the support D and that consequently a yield hinge occurs there. The conditions existing in span CD are expressed analytically as follows:

$$M_g = 15\, R_D = M_3 \quad \text{(23)}$$

$$M_c + 30\, R_D - 12\, P = 0 \quad \text{(24)}$$

$$M_c = M_2 \quad \text{(25)}$$

It follows that

$$M_3 = 4.17\, P \quad \text{(26)}$$

Assume, after applying a safety factor, that P equals 36 kips and w equals 2 kips per ft. The limit moments are then $M_x = 164$ kip ft, $M_2 = 131.5$ kip ft, and $M_3 = 150$ kip ft. The section moduli for steel I-beams having a minimum yield point stress of 36,000 lb per sq in. and a shape factor of 1.15 are: $S_x = 47.6$ cu in.; $S_2 = 38.2$ cu in.; and $S_3 = 43.5$ cu in.

Jack R. Benjamin,[8] Assoc. M. ASCE.—The method of limit design of beams and frames presented suffers from several serious shortcomings. The first is one of lack of emphasis on the basic assumptions. The method-of-collapse analysis depends primarily on two factors: the stress-strain diagram and the absence of direct tension or compression in the members. If the material assumptions are satisfactory, the second assumption is more often not satisfactory. All members of frames are subject to direct tension or compression, but the usual single-story frame has only minor tensile stresses and small-to-moderate compressive stresses. Thus under certain conditions, collapse could be predicted for most one-story frames.

The method-of-collapse analysis does not apply to multiple-story bents primarily because of the compression members. Members under combined direct compressive stress and bending do not develop plastic hinges similar to those in a beam. In a beam under a concentrated load the region of plasticity is localized, and a sharp kink will occur at failure. In contrast, in a compression member the advent of plasticity involves a considerable portion of the member, and the zone of plasticity is not localized. The advent of plasticity sharply changes the action of the member, causing rapid changes in geometry that in turn cause more plasticity. This action is highly non-

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linear. The simple rectangular, mild steel section used as a beam will develop 50% higher moment as a plastic hinge than that developed at the elastic limit. However, the same section used as a compression member may show almost no load variation between the capacity at the elastic limit and the capacity at collapse. Fig. 1 should include other curves for various values of axial load other than zero. Fig. 15 will answer part of the problem. However, the solution of the section is not the same as that of the member because of the influence of the change of geometry. The member capacity can decrease while the section capacity increases. The compression member problem is not simple. The method of the authors can apply only to members without direct compressive stress.

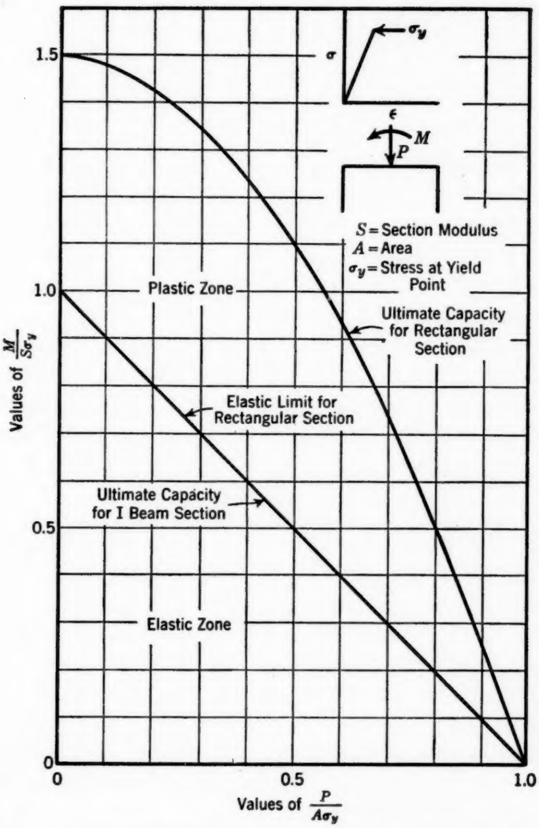

FIG. 15.—VARIATION OF MOMENT CAPACITY WITH DIRECT LOAD FOR TWO MILD STEEL SECTIONS

The second serious limitation of the method concerns the mode of failure. The failure load of a single bent, as computed by the method of this paper, may be much greater than that of a structure made up of several of these bents. The structure may fail torsionally in a lower mode than the single-bent analysis would indicate. Any analysis of an actual structure for collapse must search out the lowest failure mode. Unfortunately, torsional failures must be considered. Loadings on actual structures are far from the simple loadings used, and in almost every case torsional failure is a possibility.

A further practical problem to be solved before frame collapse can be considered a design criterion is that of loadings. The loadings used in the paper do not reflect the designer's problem. The designer must consider distributed loads of three types:

1. Dead loads whose values are fairly well known and cannot change.
2. Live loads whose service impositions on the structure are estimated by intelligent guess.

3. Transient loads (such as wind and earthquake) that are normally accounted for by using higher allowable stresses, thereby recognizing the transient nature of the loads.

The first two loading types are essentially vertical, but the latter is essentially horizontal. The paper does not indicate how these three distinctly different loadings shall be treated, although this is a fundamental design problem. The writer's studies of collapse indicate that this problem must be resolved before collapse can be considered a satisfactory design criterion. It might be mentioned also that any loading that is a function of time might momentarily give rise to collapse conditions in a structure without particularly damaging it because of the short time duration of the loading.

A second loading problem is that of the factor of safety. Should all loadings be multiplied by the same factor of safety? This course does not appear reasonable. This is a particularly knotty problem yet the paper assumes a uniform factor of safety.

The methods and ideas presented in this paper are very interesting from an academic standpoint. However, their limitations and assumptions are not fully explained. These limitations and assumptions should be underlined because design of actual structures using just these principles could be highly dangerous. Furthermore, a large number of knotty practical problems must be solved before limit analysis for collapse can be of much value.

This paper is a fine beginning toward a more rational basis for design of beams and frames, but it must be emphasized that it is just a beginning.

E. P. Popov,[9] Assoc. M. ASCE.—An outstanding addition to the methods of structural analysis was made in this paper. The method developed by the authors for establishing the upper and lower bounds for the load factor at collapse is very ingenious, and the suggested trial-and-error procedure for narrowing the bounds appears to be quite simple for many ordinary frames. The method developed is general and has already been extended to space frames.[10]

Although the problem of limit design is admirably solved from the mathematical point of view, some of the assumptions and limitations of the method are not clearly indicated, and adequate references to related work are lacking. The following remarks are intended to urge caution in the application of the method to practical problems.

The writer considers that no treatment of limit design of frames is complete without some mention of the extensive analytical and experimental work done at Cambridge University, Cambridge, England, under the direction of J. F. Baker, Assoc. M. ASCE. For example, the virtual work technique for limit design that may appear novel to some readers was described by Mr.

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[10] "The Limit Design of Space Frames," by Jacques Heyman, *Journal of Applied Mechanics*, Vol. 18, No. 2, June, 1951, p. 157.

Baker.[11] However, the application of the virtual work principle to obtain an upper bound for the safety factor is a creditable accomplishment of the authors.

In applying the limit design method to frames, several limitations must be clearly kept in mind. First, it is assumed that all compression members possess a sufficiently low slenderness ratio so that instability does not occur. Thus, while it is true that, for example, most building columns would fulfil such a condition, the instability of compression members cannot be ruled out for all conceivable cases. Second, in the "Introduction," it is stated: "The limit moments are known throughout the structure * * *"

This presumably means that the effect of axial force and shear on the limit moment of a member is known. Actually, much experimental research is needed to determine such quantities for common members. On the other hand, in the numerical example given in the paper, complete disregard of axial forces and shears found in the frame is implied. The axial forces and shear in various members of Fig. 6 and of Fig. 8 are different. Thus, from Fig. 5 it is seen that the upper member, according to the first assumption of hinges, is under no axial load, while the member AB is compressed by a force equal to $6\ M_o/h$. Similarly, by applying the conditions of equilibrium to the various members of the frame solved in Fig. 8, it may be easily shown that the axial forces vary from a tensile force equal to $3\ M_o/2\ h$ to a compressive force of $M_o/3\ h$. However, the safety of the structure is invariably based upon a common limit moment M_o alone. To be sure, Mr. Baker contends[12,13] that for small axial stress the reduction in the limit moment is surprisingly small, and thus the limit design is sufficiently developed for single story bents with light roof loads. However, the writer feels that, without further experimental verification, the method cannot be safely extended to practical cases in which large axial forces exist in the members. For example, an interaction curve for the collapse of an 8 WF 31 beam under axial loads and applied bending moments (as established in preliminary tests[14] at Lehigh University, Bethlehem, Pa.) indicates that, if there is no axial load, $M_o = 1{,}200$ in.-kips. However, if $P = 100$ kips, M_o is reduced to approximately 1,000 in.-kips. Similarly, when $P = 150$ kips, $M_o = 800$ in.-kips.

A. Hrennikoff,[15] Assoc. M. ASCE.—Presenting concisely some of the recent developments in the theory of limit design, this paper is interesting and instructive. It deals with a subject that has been commanding much attention from engineers.

Although the rise of interest in this field is comparatively new, dating from the 1939 publication of the well-known paper[16] by J. A. Van den Broek, M. ASCE (who introduced the term "limit design"), the basic idea has been used by engineers, in application to certain types of members, for a long time. Thus, the standard design of rivets, splices, and connections of tension and compression

[11] "The Design of Steel Frames," by J. F. Baker, *The Structural Engineer*, Vol. XXVII, 1949, p. 397.

[12] "The Design of Steel Frames," by J. F. Baker, *The Structural Engineer*, Vol. XXVII, 1949, p. 408.

[13] *Ibid.*, p. 428.

[14] "Tests of Columns Under Combined Thrust and Moment," by Lynn S. Beedle, Joseph A. Ready, and Bruce G. Johnston, *Proceedings of the Society for Experimental Stress Analysis*, Vol. VIII, 1950, p. 109.

[15] Prof., Civ. Eng., Univ. of British Columbia, Vancouver, B. C., Canada.

[16] "Theory of Limit Design," by J. A. Van den Broek, *Transactions*, ASCE, Vol. 105, 1940, p. 638.

steel members is nothing but limit design, since different parts of the members involved (such as rivets and sections of the main material and splice plates) are arbitrarily assigned capacity loads fully consistent with statics, so that no part is overstressed, while at the same time no cognizance is taken of the deformation of different parts. This established method was devised for the design of connections out of sheer necessity, since the problem was found too difficult for the elastic theory. The method proved to be safe largely because of the ample ductility of structural steel.

Mr. Van den Broek's theory of limit design and the authors' elaboration of it projects the same principle (assigning capacity values to the members) into the realm of complete statically indeterminate flexural structures consisting of continuous beams and rigid frames. The element of compelling necessity is lacking in this field, since continuous beams and rigid frames devoid of side sway can be analyzed conveniently by moment distribution and other methods of elastic theory, and frames subject to side sway can be analyzed by some adaptation of statics, such as the portal and cantilever methods. Nevertheless, the method of limit design offers some economy compared to the other methods and for this reason is fully justifiable. At the same time, in view of its less conservative nature, its adoption should be preceded by a searching investigation of underlying assumptions. It is gratifying, therefore, to see in this paper formal rigorous proofs of the statements establishing the upper and lower limits of the factors of safety in the structure from the viewpoint of limit design.

There are, however, some other important considerations which have not received due attention. One of these is the question of plastic hinges. In the opinion of the authors, such hinges develop freely at several sections of the structure and provide all the angle changes necessary for consistency of deformation that takes place prior to the collapse of the structure. It is tacitly assumed here that the structure cannot fail by actual physical breaking at some of the earlier formed hinges before the limiting state is reached and the structure begins to act as a mechanism. The basis for this unstated assumption apparently lies in the shape of the assumed (M-ϕ)-curve (Fig. 1), in which the limiting moment M_o continues constant indefinitely as the angle change ϕ continues to increase. However, the absolute value of ϕ that a beam can sustain before breaking depends on the length l, over which the limiting moment is acting. If this length is small, the maximum possible angle change is small too, in spite of a long horizontal part of (M-ϕ)-curve in Fig. 1. For example, a 1-ft length of an 18-in. I-beam can easily have its flanges deformed as much as 1 in. each, and can thereby undergo (without ill effects) an angle change of approximately 6°, but the same beam, 1 in. long, will probably break before bending through the same angle of 6°, because such bending would mean doubling the length of the tension flange and compressing the other flange to a length of zero. An impossibility of such a failure at the plastic hinge must be firmly established before the theory of limit design can be accepted, and the conditions that would prevent hinge failure must be established definitely.

One of the main factors that determine whether the beam should fail in this manner is the shape of the (M-ϕ)-curve beyond the elastic stage. Assume first an extreme case in which the (M-ϕ)-curve consists of two straight lines (Fig.

16), the elastic part OA and the plastic part AB, and consider an example of bending of a fixed-end beam under a gradually increasing uniform loading w (Fig. 17). As long as the elastic range is not exceeded, the moment curve is a parabola with the negative end ordinates two times greater than the positive center ordinate. When the load intensity reaches the value w_e, the end moments just reach the limit value M_o—in other words $M_o = \frac{w_e L^2}{12}$, and the moment diagram assumes the shape FDHEG. To this point all angle changes have been elastic. If, now, w exceeds w_e, plastic hinges form at the ends, preserving the same values of moments there, whereas the positive center moments gradually increase beyond $\frac{w_e L^2}{24}$ up to the limiting value $\frac{w_e L^2}{12}$, and the moment diagram approaches the limit curve FD_1KE_1G.

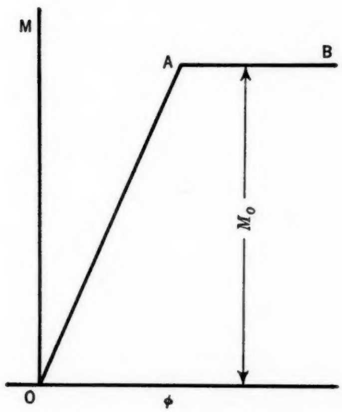

Fig. 16

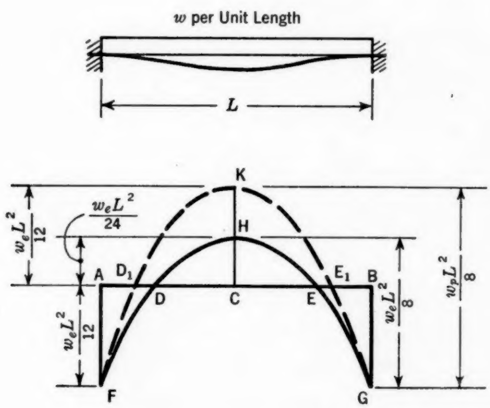

Fig. 17

It must be remembered that the total angle change on one half the length of the beam AC must be zero both in the elastic and plastic stages, as demanded by deformation of the fixed-ended beam. At the end of the elastic behavior the positive angle change on the length DC equals numerically the negative angle change on the length AD, both these angle changes being proportional to the corresponding areas under the moment curve. Toward the limiting condition corresponding to the load w_p, the still-elastic moment at the center, doubles its value compared to the value under the load w_e, and the length of the positive moment region D_1C gets greater than DC. These circumstances combine to yield a positive angle change that is more than twice the magnitude of the one created by load w_e. However, the region of negative moments shrinks to the length AD_1, and the negative bending moments are everywhere smaller in value than under the load w_e, except at one point A, where the moment remains the same. This means that all increase in the negative angle change necessary to balance the increased positive angle change and to compensate for the loss of some negative angle change must take place at one section—that is, on an in-

finitesimal length at point A, Fig. 17. Since such a concentration of angle change would result in infinite strains, the beam is likely to fail at the ends before the limit condition visualized in limit design is materialized.

Assume now a more general case of the (M-ϕ)-curve in Fig. 18, and let ϕ represent from now on an angle change occurring on a unit length of the beam

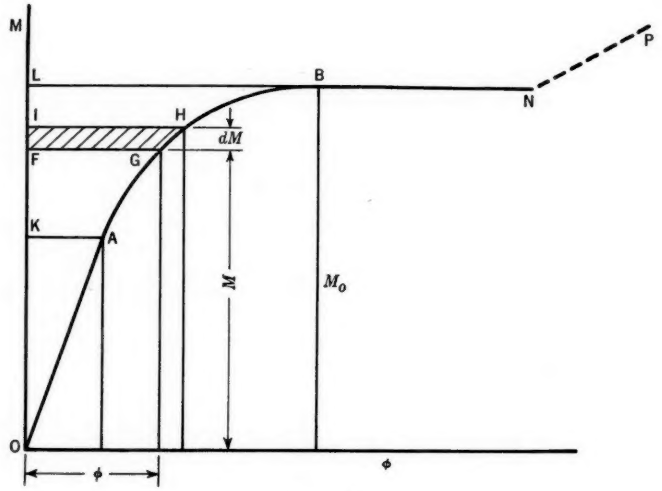

FIG. 18

corresponding to the value of the bending moment M. An increment of moment dM occurs on the length of the beam dl determined by the value of the shearing force V in this manner: $dl = \frac{dM}{V}$. The angle change on this length is evidently equal to

$$\phi\, dl = \frac{\phi\, dM}{V} \quad \text{.......................} (27)$$

In Eq. 27 $\phi\, dM$ represents the area of the horizontal strip FGHI, Fig. 18. By summation of these strips the total angle change between any two points on the curve, such as points A and B (assuming the shearing force V constant in the interval), is found to be:

$$\phi = \int_A^B \phi\, dl = \frac{\int_A^B \phi\, dM}{V} = \frac{\text{area } \overline{\text{AKLB}}}{V} \quad \text{...........} (28)$$

The area in the numerator of Eq. 28 is bounded by the axis of M, the (M-ϕ)-curve, and the two abscissas, corresponding to the points on the curve between which the angle change is evaluated. It is interesting to note that, with the (M-ϕ)-curve extending horizontally without limit, no contribution to the angle change is provided by the horizontal or plastic part of the curve; because advancing the point B along the curve to the right will not increase the

area AKLB. Thus, contrary to intuition, the plastic part of the (M-ϕ)-curve proves incapable of contributing to the necessary angle change at the plastic hinge and to preclude failure of the structure there while other plastic hinges are developing.

The situation is different if the (M-ϕ)-curve exhibits a strain hardening part, indicated in Fig. 18 by a dotted line NP. When a stress condition in the beam extends into the strain hardening region, the area in the numerator of Eq. 28 begins to increase again, thus providing the necessary angle change at the hinge. The presence of a plastic region prior to strain hardening increases the referred-to area of the (M-ϕ)-curve, and thus decreases the necessary rise of moment along the strain hardening part, thus assisting in the formation of the hinge.

As follows from Eq. 28, the angle change occurring in a certain range of moments is inversely proportional to the shearing force.

An exact solution of the problem of bending of a typical fixed-end uniformly loaded **I**-beam, made of typical mild steel,[17] shows that the ends of the beam would extend into the strain hardening region even before the limit of proportionality is reached at the center.

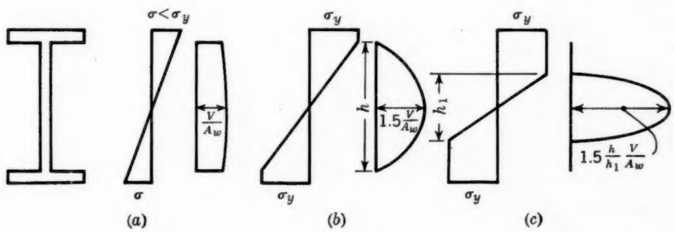

FIG. 19

In the case of **I**-beams and channels, there is another factor which interferes with the behavior of beams not subject to strain hardening as visualized in the theory of limit design. The **I**-beams are characterized as a rule by appreciable shearing stresses in the web. In the elastic range these stresses are roughly uniform and equal to $\frac{V}{A_w}$ (Fig. 19(*a*)) where A_w is the area of the web. When the flanges have just reached the yield point, the shearing stresses follow a parabola with the greatest ordinate $1.5\,\frac{V}{A_w}$ at the center. As yielding approaches the center, the shearing stresses concentrate near the center (Fig. 19(*c*)) increasing in the ratio h/h_1, compared to the condition shown in Fig. 19(*b*). These deductions follow from a consideration of the static equilibrium of an element of beam, as used in the elementary determination of shearing stresses. When the depth h_1, over which the normal stress in the web of the beam is still elastic, becomes quite small, the shear stress at the center becomes excessive. In the absence of strain hardening this condition cannot be relieved by warping

[17] "Theory of Inelastic Bending With Reference to Limit Design," by Alexander Hrennikoff, *Transactions*, ASCE, Vol. 113, 1948, p. 235.

of the section, and shear failure must necessarily ensue. Thus, an I-beam with the $(M\text{-}\phi)$-characteristic assumed in Fig. 1, is incapable of sustaining any shearing force at and near its horizontal plastic section, unless strain hardening comes in and redistributes the shearing stress again over the entire section of the web.

Thus, the major characteristic of the beam essential for formation of plastic hinges is the strain hardening rather than the plastic stage although the latter is very valuable in keeping down the hinge moments in beams subject to strain hardening. Without strain hardening the beams would fail at the values of the load below the limit load.

Most of the ductile metals are subject to strain hardening and thus may be suitable for limit design; but the moments present at plastic hinges are unknown beforehand. As the hinges form and extend, the hinge moments grow larger. The moments are greater at the earlier formed hinges than at the later ones. This complicates the application of the theory of limit design considerably. It is possible that an assumption of constant moment values is still admissible, but the limits of errors involved in it should be investigated.

The degree of strain hardening necessary to avoid an early failure becomes an important question. Most ductile metals are likely to have the necessary requisites, but some aluminum and steel alloys are probably unfit for limit design.

An exact theory of bending beyond the elastic range is needed to answer some of these questions. Some investigations have been made along these lines.[17] Further extension of this theory, allowing for the effect of shear and for the warping of the section, is now in order.

In conclusion, the writer wishes to state that the purpose of his discussion has been not to disparage or to minimize the excellent work performed by the authors, but to state the limitations of the theory and to point to some comparatively unexplored but basic phases of the problem, which must be examined and settled prior to further advances along the road taken by most of the investigators in the field of limit design.

P. S. Symonds[18].—The authors have rendered an invaluable service in stating and proving in rigorous terms two theorems that are basic in the plastic methods of analysis and design of beams and frames. They have filled a serious gap in the theory. Since they were concerned with basic theory, they could not elaborate on the physical significance or justification of the underlying hypotheses, nor could they deal at any length with practical applications.

"Limit moment" has also been termed the "fully plastic moment" of a given section since it represents the limiting condition in a mild steel beam when plastic zones have spread all the way across the section except for a zone of infinitesimal thickness in which the stress changes from the yield stress in tension to that in compression. As the fully plastic moment is approached, the curvature at a given section tends toward infinity. Thus, when the fully

[18]Associate Prof. of Eng., Brown Univ., Providence, R. I.

plastic moment is reached, by hypothesis a finite change of slope in an infinitesimal distance can occur. At such a section free relative rotation would take place under constant moment, that is, as if a "plastic hinge" were inserted.

These notions of fully plastic moments and plastic hinges represent idealized physical quantities since they are never strictly realized in physical structures. For example, the relative rotations at a hypothetical plastic hinge imply infinite curvatures and hence infinite strains; this means that the material actually enters the strain hardening range, and the actual moment must rise above the theoretical limit moment. More importantly, the fully plastic moments in the authors' theory and, in general, in the simple plastic methods of analysis and design, are treated as constants for a given section and material. In fact, they are always functions of local loading conditions such as the existence of transverse shear or axial force, and depend upon the rate of loading. Moreover, effects of changes of shape are neglected, and these may be important when large axial forces are present.

The basic hypotheses must be regarded as semi-empirical postulates whose utility and justification must be verified by tests on actual structures. Much of the needed experimental corroboration has been supplied by H. Maier-Leibnitz[19] in his tests on continuous beams and by J. F. Baker,[20] Assoc. M. ASCE, in tests on small portal frames. More work is needed, especially on frames of full-scale size and standard commercial members, but the work already done is strong evidence that the plastic theory based on the hypotheses of the paper is not only simple and direct, but leads to the prediction of failure loads in reasonably close correspondence with observed loads at which the rate of increase of deflection with load rises sharply to excessive values. Mr. Baker[21] and his associates have also shown clearly the potential practical advantages of the plastic methods, in suitable problems, over conventional elastic methods.

An alternative method for the calculation of plastic failure loads may be based directly on the authors' theorem that if plastic hinges are inserted at sufficient sections of the frame to render it a kinematic mechanism, then the corresponding load is always either greater than, or equal to, the actual collapse load of the frame. It leads to especially rapid solutions for rectangular frames and would seem to make possible the determination of the plastic collapse load of a multi-story, multi-bay frame, or of a shed-type frame in a surprisingly short time.

Considering the two-story frame treated as an example by the authors, any mechanism, including the actual one, can be regarded as built of combinations of certain elementary mechanisms. In any problem there are three types of elementary mechanisms, namely: those associated with failure of single beams under their own loads, failure of individual panels or stories of frame, and "failure" (in a special sense) corresponding to rotation of a joint at which three or more members are connected.

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[19] "Test Results, their Interpretation and Application," by H. Maier-Leibnitz, *Preliminary Publications*, 2nd Congress, International Association for Bridge and Structural Engineering, Berlin, Germany, 1936, p. 97.

[20] "A Review of Recent Investigations into the Behaviour of Steel Frames in the Plastic Range," by John Fleetwood Baker, *Journal of the Institution of Civil Engineers*, London, England, Vol. 31, 1949, p. 188.

[21] "The Design of Steel Frames," by J. F. Baker, *The Structural Engineer*, London, England, Vol. 27, 1949, p. 397.

Fig. 20 shows these three types of elementary mechanisms for the authors' frame and loading. The dots represent plastic hinges, and angle θ in each case is arbitrary but small. Joint mechanisms (Fig. 21(*c*)) are devices for representing arbitrary rotations and need not be true mechanisms.

There are always just as many of these elementary mechanisms as there are independent equations of equilibrium relating the moments which must be considered in the plastic collapse analysis; thus in the authors' example there are six elementary mechanisms. These can be combined with each other just as can equations of equilibrium. The actual mechanism is either one of these elementary mechanisms or the particular combination of them that has the lowest corresponding load of any possible combination. The guide in combining them is the elimination of plastic hinges in the mechanisms being combined. In practice it is easy to see what combinations are likely to be advantageous. Furthermore, when the principle of virtual work is used, as in the paper, it is a matter of only a minute or two to compute the failure load corresponding to any combination.

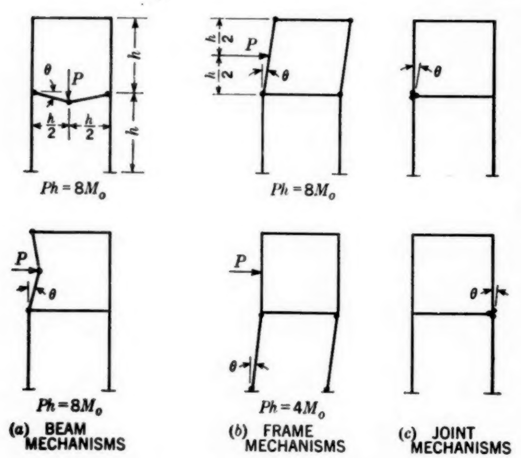

Fig. 20.—Elementary Mechanisms

Fig. 21 shows a number of possibly advantageous combinations. The correct combination is that of Fig. 21(*c*) which can be thought of as composed of the lower of the mechanisms of Fig. 20(*a*) and those of Fig. 20(*b*), with the angle θ in each diagram chosen so as to eliminate the plastic hinge at the top left hand joint. The test for correctness is that no other elementary mechanism can be combined with that of Fig. 21(*c*) to give a new mechanism with a lower collapse load than 11/3 M_o/h. With a little practice it is not difficult for the analyst to see when all possibly advantageous mechanisms have been tried. There is then no need of making any elastic calculations or of seeking a lower bound, and this is often a major practical advantage of the method. A more complete account is to be published elsewhere.[22]

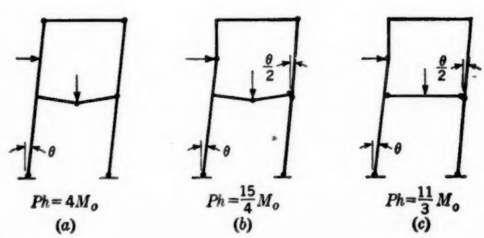

Fig. 21.—Advantageous Combinations of Elementary Mechanisms with Corresponding Failure Loads

[22] "The Rapid Calculation of the Plastic Collapse Load for a Framed Structure," by B. G. Neal and P. S. Symonds, *Inst. of Civ. Engrs.*, London, England (publication pending).

This technique (with different physical concepts) has also been used successfully in the determination of shakedown load limits of continuous frames,[23] that is, of the load limits within which, no matter in what order or how many times a set of loads are applied, plastic flow ultimately ceases. This general problem always includes, as a special case, the problem of failure under a single application of a set of loads. Its practical importance lies in the fact that when loads are repeatedly applied in certain sequences, failure of the type considered by the authors may occur at lower load values than are required for failure under a single load application. The authors' theorems apply only in this rather restricted problem.

Finally, the writer would raise a question of terminology. The term safety factor as used in this paper seems unfortunate since it differs from ordinary structural usage. The method of the paper and others available really determines the failure load of a given frame not a safety factor in the usual sense. The latter is a number chosen in advance by the designer, for instance, 2.0 or 1.75, that depends on his estimate of uncertainties of materials, fabrication methods, loading, and other quantities in a given type of structure. The structure is so designed that the given working loads, when multiplied by this factor, are just capable of producing failure. Safety factor as used in the paper has no relation to the one defined and would seem to make for some unnecessary confusion by conflicting with ordinary terminology.

H. TACHAU,[24] JUN. ASCE.—A valuable contribution to the theory of limit design has been made by the authors of this paper. Because it is clearly presented and based on principles familiar to structural engineers, the new procedure is easy to learn and easy to apply. The proposed methods make it a relatively simple matter to analyze the collapse of rigid frames with several bays and several stories. Thus, complex frames can now be subjected to limit design, whereas this was not practically feasible previously.

Certainly, this procedure is much shorter than the conventional analysis, and it would be of great benefit to the analyst if the new theory could eventually supercede elastic methods. However, there still exists a gap between the theory as presented in the paper and its application to the actual design of framed structures. Since the authors' efforts were evidently concentrated on the establishment and proofs of the new theorems, it could hardly be expected that they would cover this aspect in the short space allotted to one paper. Part of this discussion is intended to define that gap in more detail.

By basing their theory on a bending moment curvature diagram (Fig. 1), the authors have not encountered the difficulty of relating the limit moment to the stress-strain curve of the material (usually obtained from a standard tension test). The main purpose of Fig. 1 is to establish a working value for the limit moment M_o. Suppose all members of a structure have the same cross section at the critical locations where yield hinges will occur. At first it may appear that in this particular case M_o would be the same at all sections

[23] "Recent Progress in the Plastic Methods of Structural Analysis," by P. S. Symonds and B. G. Neal, *Journal of the Franklin Institute* (publication pending).

[24] Asst. Prof. of Civ. Eng. and Mechanics and Hydraulics, State Univ. of Iowa, Iowa City, Iowa.

involved. On second thought, however, this may not be strictly true as there are a number of factors influencing the value of the limit moment M_o.

In a general way the following variables may have to be considered in establishing bending moment curvature curves for a given material: (1) Geometric shape of the member, which involves the shape of cross section and also changes along the axis of the member; (2) distribution of bending moments along the member; and (3) presence of axial and shear stresses in addition to the bending moments.

To be more specific, consider a frame of structural steel. Apparently, at the present time (1951) limit moments are not readily available for structural sections of various sizes. However, as a crude approximation, limit moments for wide-flange beams can be based on the flexure formula, using the ultimate strength in place of the working stress. For other sections limit moments can often be approximated by assuming the entire cross section to be subjected to its ultimate strength. More exact values of M_o can be calculated by a method developed by Mr. Hrennikoff.[25] This method is applicable to prismatic members symmetrical about their neutral axes. But many steel beams are composed of relatively thin parts, and while local crinkling of the compression flange is not usually a factor in elastic analysis, it might be of importance after yielding has begun. Likewise, lateral instability is of importance for many economical steel sections, especially wide-flange beams. The latter topic has been studied in connection with the conventional elastic analysis[26] and is incorporated in the present American Institute of Steel Construction (AISC) design specification for steel buildings. Possibly the formula for elastic instability can be modified so as to apply to limit design. An investigation of lateral instability in the plastic range has been published,[27] but the results of this research are applicable to beams of rectangular cross section only.

To obtain rigid connections between members, the joints are generally reinforced with plates, seat angles, or brackets. Thus, the bending resistance right at the joint will be higher than elsewhere, and the critical sections will occur at some distance away from the connection. Obviously this effect is important for short members only.

In addition, other less important factors may have to be considered under certain circumstances, such as lack of uniformity of the material (which may become important in cases where the bending moment varies gradually, as, for example, on the right side of the lower horizontal member in Fig. 22, or for members of variable cross section), duration of loading, rate of load application, nonsymmetrical cross sections, and the restraining actions of web stiffeners (at concentrated load points).

In reinforced concrete frames all joints are rigid by the very nature of the construction. Limit design concepts have been applied to concrete beams

[25] "Theory of Inelastic Bending with Reference to Limit Design," by Alexander Hrennikoff, *Transactions*, ASCE, Vol. 113, 1948, p. 213.

[26] "Strength of Beams as Determined by Lateral Buckling," by Karl deVries, *Transactions*, ASCE, Vol. 112, 1947, p. 1245

[27] "The Lateral Instability of Yielded Mild Steel Beams of Rectangular Cross-Section," by B. G. Neal, *Philosophical Transactions of the Royal Society of London*, Series A, Vol. 242, No. 846, 1950, pp. 197–242.

and columns[28] but, to the knowledge of the writer there is no published record of application to concrete frames. Very few load-deflection curves or similar experimental data are to be found in the literature, but the deflections at the "yield" and ultimate can be calculated,[29] and in some cases the moment curvature relation will be of the type shown in Fig. 1. The main difficulty in applying the method of the paper to concrete design is the necessity of placing the reinforcing symmetrically about the bending axis. Otherwise, M_o will be different for positive and negative bending. Therefore, it would be worthwhile to try extending the new method to cases in which the positive limit moment is different from the negative limit moment.

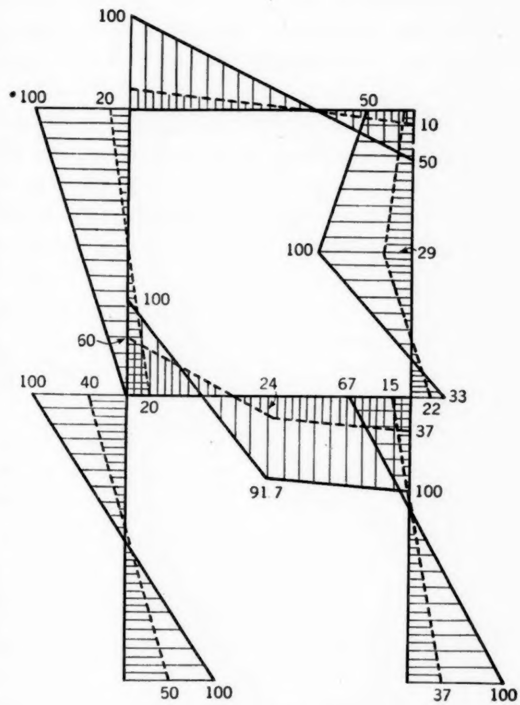

FIG. 22

The two-story frame described in Figs. 2 to 8 serves as an excellent medium for demonstrating the new method. The selection of the location of the yield hinges is not at all difficult. They will be at joints and corners and under concentrated loads. If the number of yield hinges is less than the number of redundants, the equations of statics will not be sufficient to solve the forces on all members. But, in such instances arbitrary values may be assigned to these forces, so long as they are statically compatible with the applied loads. It can probably be proved that the number of yield hinges required for over-all collapse must be at least equal to one more than the number of redundants.[30] On the other hand, the hinges could be arranged so the movements of the members are not uniquely defined. There is no obvious reason for deciding whether or not the mechanism created by the introduction of yield hinges should be such that the motion of each link is completely controlled by and depends on the motions and positions of each of the other links.

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[28] "The Design of Reinforced Concrete Structures," by Dean Peabody, Jr., 2d Ed., John Wiley & Sons, Inc., New York, N. Y., 1946.

[29] "Ultimate Strength of Reinforced Concrete Beams as Related to the Plasticity Ratio of Concrete," by Vernon P. Jensen, *Bulletin No. 345*, University of Illinois Engineering Experiment Station, Vol. 37, 1943, p. 35.

[30] "Theory of Limit Design," by J. A. Van den Broek, John Wiley & Sons, Inc., New York, N. Y., 1948, p. 32.

As the frame is subjected to gradually increasing forces the moments grow beyond the elastic limit, as shown by Fig. 22. In this figure, the bending moments based on the elastic distribution are represented by dotted lines and figures, and the distribution at collapse is shown by solid lines. The limit moments M_o are assumed as 100, and the elastic distribution has a maximum value of 60. The two arbitrary values of 60 and 100 represent a reasonable relation for a wide-flange beam. The similarity between the shape of the moment diagrams at the elastic limit and at collapse is quite striking and lends further support to the validity of the proposed method. The highest elastic moment occurs at the left end of the lower horizontal member. This same member is also subjected to high shearing forces and this is where inelastic action would first begin. Ordinarily, this place should be made stronger than the rest of the structure. At the same time, the left column is very highly

TABLE 1.—LIMIT DESIGN OF BENT OF FIG. 22

Member	Elastic Limit[a]			Total Collapse[a]		
	Maximum Moment	Axial Stress	Maximum Shear	Maximum Moment	Axial Stress	Maximum Shear
Lower left vertical	50	−198	90	100	−533	200
Upper left vertical	20	−29	39	100	−150	100
Lower right vertical	37	+55	52	100	+167	167
Upper right vertical	29	+29	103	100	+150	267
Lower horizontal	60	−51	168	100	−100	383
Upper horizontal	20	−39	29	100	−100	150

[a] Tension is designated +, Compression −.

stressed at the fixed end: the moment is 50, the axial compression is 198, and the shear has a value of 90 at this point. All forces are expressed in terms of 100 M_o/h. For comparison see Table 1.) The elastic moment of 60 is reached when $P = 142.4$, but collapse occurs only when P reaches 367. So the margin of safety is roughly 150% based on the elastic limit. Of course, the amount of the increase depends on the shape and other variables mentioned previously.

After the distribution of moments at collapse has been determined, the axial and shear forces can be found for each member. If they reduce M_o, the critical section may be increased in size, and if necessary, the previous analysis can be reviewed. Maximum moments, axial stresses, and maximum shears for each member of the bent in Fig. 3 are summarized in Table 1.

At failure, the right-hand portion of the lower horizontal in Fig. 3 is subjected to moments varying from M_o at the right end to 0.917 M_o at the center. Though the original hypothesis that yield hinges occur at discrete locations has not been strictly violated, it is doubtful if this condition could be approached. Almost half the member is in a plastic condition, and collapse might occur at a lower load than required by the mechanism of Fig. 7.

VINCENZO FRANCIOSI.[31]—The elegant method of calculation of the safety factor introduced in this paper permits the neglect of all that happens between the first appearance of the plastic phenomena and the total collapse, since it considers the structure only in this final phase. It does not consider, however, the effects of the axial force that may, in some cases, play a very important role. Owing to the presence of the axial force, in fact, both the value of the moment corresponding to the beginning of the plastic phenomenon and the value of the limit moment M_o decrease. The point of interest, as was stated in the Introduction, is not to know the relation between the bending moment and the curvature in the elastic-plastic state but the new reduced value M'_o of the limit moment.

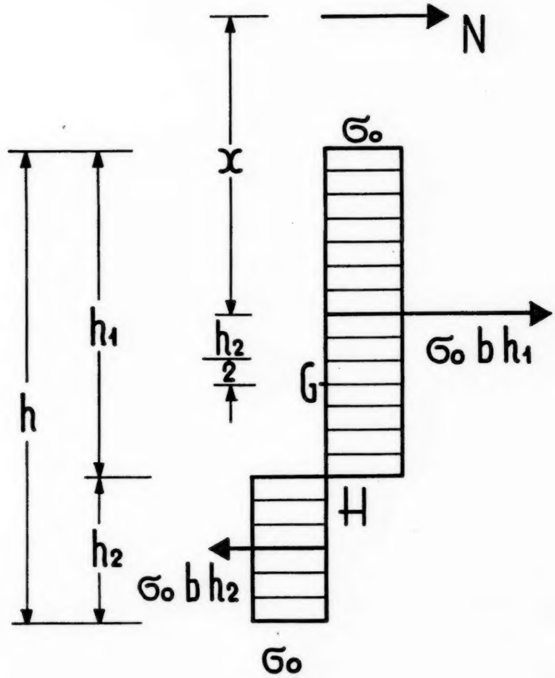

FIG. 23

For the calculation of the reduced limit moment the rectangular cross section that is the most frequently met with in the technical practice is used. The diagram of the stresses originated by an axial force N and by the limit moment reduced to M'_o is shown in Fig. 23. The conditions of equilibrium to a translation along the beam axis give

$$\left.\begin{matrix} h_1 \\ \\ h_2 \end{matrix}\right\} = \frac{h}{2}\,\frac{N_o \pm N}{N_o} \qquad (28)$$

[31] Prof. of Construction Science, Univ. of Naples, Naples, Italy.

in which $N_o = \sigma_o b\, h$ (σ_o being the yield stress) equals the normal force that would determine by its sole action the yielding of the section. The condition of equilibrium to rotation gives the value

$$x = \frac{N_o}{N}\frac{h_2}{2} \qquad (29)$$

Thus

$$M'_o = N\left(x + \frac{h_2}{2}\right) = \frac{N_o h}{4} = \left(1 - \frac{N^2}{N^2_o}\right) \qquad (30)$$

Since[32]

$$\frac{N_o h}{4} = \frac{\sigma_o\, b\, h^2}{4} = M_o \qquad (31)$$

$$M'_o = M_o\left(1 - \frac{N^2}{N^2_o}\right) \qquad (32)$$

If the forces N are known, from Eq. 32 the value of the reduced limit moments M'_o is drawn, and the calculation of the safety factor goes on by means parallel to those proposed in the paper. This procedure is used, for instance, in the case of a deflected beam loaded with an axial constant force.

Generally, however (as, for instance, in the case of the frames), the axial forces N depend on the loads applied, and in the collapse phase they are linear homogeneous functions of the couples M'_{oi} applied at the yield hinges.

Consider, for instance, the strucutre shown in Fig. 24. In the line C D the shearing (constant) force τ is given by

$$\tau = \frac{M'_{o5} + M'_{o3} - M'_{o1}}{h} \qquad (33)$$

in which M'_{oi} is the reduced limit moment relative to the ith bar. (Such moment has one value along the whole bar, as the axial force it contains is constant, and the section is presumed constant.) For the equilibrium of the line D E the axial force in the Cols. 4 and 5 (tension forces considered positive) is

$$-N_4 = N_5 = \frac{N'_{o1} - M'_{o3} + 3\, M'_{o5}}{2\, h} \qquad (34)$$

For the equilibrium in the line A B the shearing force is

$$\tau = 2\,\frac{M'_{oi}}{h} \qquad (35)$$

For the equilibrium of the line E G the value of n is

$$n = \frac{M'_{o1} + \frac{2}{3}\, M'_{o2} + M'_{o3} + M'_{o5}}{P\, h} \qquad (36)$$

[32] "Scienza delle Costruzioni," by A. Galli, Pellerano, Naples, Vol. 2, 1951, p. 216.

Studying finally the beam C F two reactions are obtained

$$R_c = \frac{n\,P}{2} + \frac{2\,M'_{o3}}{h} \dots\dots\dots\dots\dots\dots\dots (37a)$$

$$R_f = \frac{n\,P}{2} - \frac{2\,M'_{o3}}{h} \dots\dots\dots\dots\dots\dots\dots (37b)$$

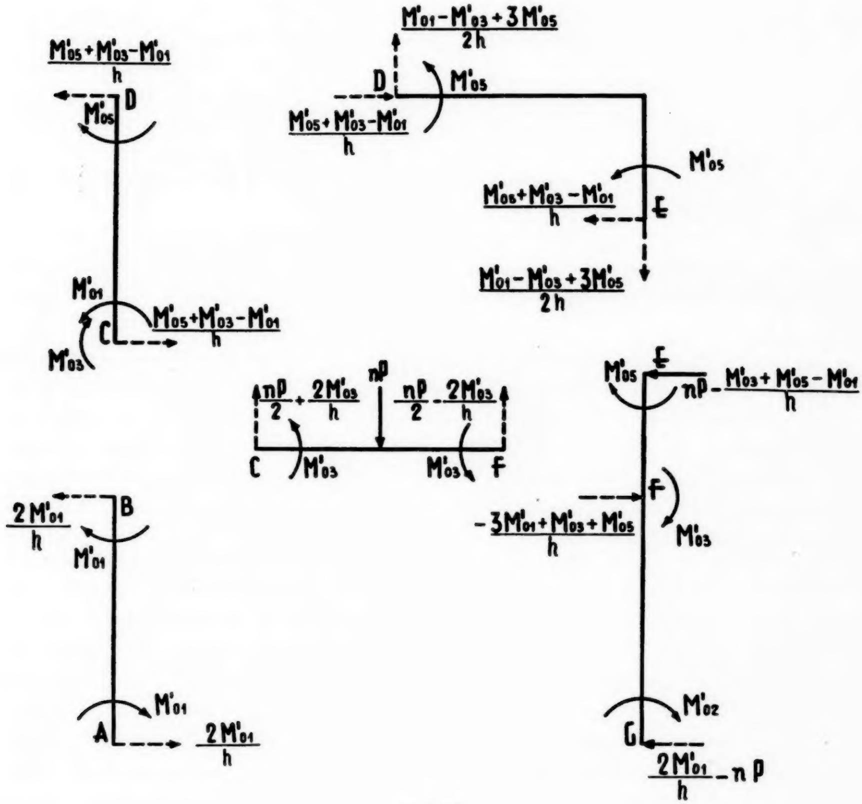

FIG. 24

which give the normal forces

$$N_1 = N_4 - \frac{n\,P}{2} - \frac{2\,M'_{o3}}{h} \dots\dots\dots\dots\dots\dots (38a)$$

$$N_2 = -\,N_1 - n\,P \dots\dots\dots\dots\dots\dots\dots (38b)$$

The force N_3, in its turn, is given by the difference of the shearing forces in lines A B and C D

$$N_3 = -\,\frac{2\,M'_{o1}}{h} + \frac{M'_{o5} + M'_{o3} - M'_{o1}}{h} \dots\dots\dots\dots (39)$$

The value of force N_6 is of no interest as no yield hinge is situated in bar 6. Eqs. 34, 38, and 39 give the values of the normal forces as liner homogeneous functions of the couples M'_{oi}. Therefore, for a generic frame,

$$N_i = \sum_1^n{}_j a_{ij} M'_{oj} \; (i = 1, 2, \cdots n) \ldots\ldots\ldots\ldots\ldots\ldots (40)$$

in which n is the number of the bars of the frame on which some yield hinges are situated.

By means of Eq. 32 the system

$$N_i = A_i + \sum_1^n{}_j A_{ij} N^2_j \; (i = 1, 2, \cdots n) \ldots\ldots\ldots\ldots\ldots (41)$$

of n quadratic nonhomogeneous equations in which the rectangular terms are absent is arrived at.

Having thus obtained the values of N_i and, therefore, those of the reduced M'_{oi}, the next step is either to ascertain whether the coefficient n is the safety factor or to define the interval in which said factor is contained.

H. J. Greenberg[33] and W. Prager,[34] M. ASCE.—Mr. Grinter's intuitive approach to the main theorems of the paper will doubtless be welcomed by many readers because it makes these theorems plausible with a minimum of mathematics. It is even possible that some readers would be willing to settle for such a plausibility argument reinforced by the authoritative statement that the theorems can be proved rigorously. Others, however, will insist on seeing an actual proof before they feel justified in applying these theorems. For the following reasons, Mr. Grinter's argument does not constitute such a proof. First of all, the assumption that a frame with $(n-1)$ redundancies will collapse only after n plastic hinges have developed is erroneous because it excludes the possibility of local collapse discussed in Section 5 of the paper. The assumption that the choice of a collapse mode leads to statically determinate bending moments likewise overlooks the possibility of local collapse. Next, Mr. Grinter's way of determining the necessary hardening x from the supposedly unique bending moments associated with the assumed collapse mode effectively uses the writers' Theorem I which Mr. Grinter is about to prove. Since this theorem is not yet shown to be valid at this stage, the only way of modifying the frame in such a manner that the assumed collapse mode cannot fail to be the actual one is to leave the limit moments unchanged at the chosen yield hinges and increase them indefinitely everywhere else, as is done in the last paragraph of the paper. This corresponds to $x = \infty$; therefore, the correct form of Mr. Grinter's argument furnishes the trivial lower bound zero for the collapse load. Finally, even though the assumed collapse mode becomes the actual one when the limit moments of all cross sections, excepting the chosen yield hinges, are

[33] Associate Prof. of Mathematics, Carnegie Inst. of Technology, Pittsburgh, Pa.
[34] Prof. of Applied Mechanics, Brown University, Providence R. I.

increased indefinitely, the conclusion that the collapse load obtained from this mode is an upper bound for the collapse load of the considered frame is based on the tacit assumption that Theorem V is valid. Actually, this theorem is is a consequence of Theorem I which Mr. Grinter has not yet established at this stage.

The writers agree fully with Mr. Grinter's remark that good engineering judgement is needed if the bounds obtained from the two principles are to bracket the collapse load closely. Since even the best engineering judgement may fail in the case of highly indeterminate structures of unconventional design, iterative methods providing a systematic narrowing of the gap between the bounds are needed. In this connection the work of B. G. Neal and P. S. Symonds[35,20,21] and W. Nachbar and J. Heyman[36] must be mentioned. The methods developed by these authors are based on the theorems presented in this paper and go far toward eliminating the need for engineering judgement in the determination of collapse loads and modes.

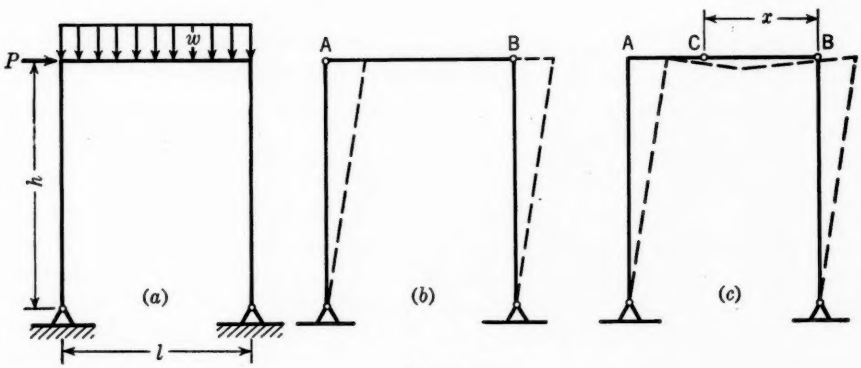

FIG. 25

Mr. Silverman's remarks reveal the fact that the term "limit design," as currently used, is misleading and should be replaced by the term "limit analysis" introduced in 1951 by W. Prager and P. G. Hodge, Jr.[37,38] Mr. Silverman's contribution to the discussion contains an example of true limit design; other examples have been presented by Mr. Heyman.[39]

As stated by Mr. Benjamin, a common safety factor applied to various types of loads is not realistic. The writers were well aware of this fact but did not think it advisable to obscure the exposition of the basic principles by the introduction of the numerous complicating conditions encountered in practice. To illustrate the application of the method to the case of concen-

[35] "The Calculation of Collapse Loads for Framed Structures," by B. G. Neal and P. S. Symonds, *Journal of the Institution of Civil Engineers*, London, England, Vol. 251, 1951.

[36] "Approximate Methods in the Limit Design of Structures," by J. Heyman and W. Nachbar, *Proceedings*, First U. S. National Congress for Applied Mechanics, Chicago, Ill. (publication pending).

[37] "Theory of Perfectly Plastic Solids," by W. Prager and P. G. Hodge, Jr., John Wiley & Sons, Inc., New York, N. Y., 1951 Chapter VII, Sec. 33.

[38] *Ibid.*, Chapter VIII, Sec. 39.

[39] "Plastic Design of Beams and Plane Frames for Minimum Material Consumption," by J. Heyman, *Quarterly of Applied Mathematics*, Vol. 8, 1951, p. 373.

trated or distributed loads which vary independently, consider the portal frame with hinged feet shown in Fig. 25(*a*). Applied to the collapse modes shown in Figs. 25(*b*) and 25(*c*), the principle of virtual work yields the relations:

$$P\,h = 2\,M_o \qquad (42)$$

and

$$P\,h - w\,l\,\frac{l-x}{2} = 2\,M_o\,\frac{l}{x} \qquad (43)$$

in which the limit moment is assumed to have the same value M_o for all members of the frame. The discussion of these relations is facilitated by the use of Fig. 26 in which $\frac{P\,h}{M_o}$ and $\frac{w\,l^2}{8\,M_o}$ are used as coordinates. Eq. 42 represents the straight line DE. According to Theorem III, the value of P obtained from Eq. 42 is an upper bound of the collapse load. This means that the points to the right of the line DE in Fig. 26 cannot represent safe states of loading. According to the ratio of P and w, the hinge C may develop at any point between the corner A and the center of the span AB (Fig. 25(*c*). Therefore, Eq. 43 must be discussed for $\frac{l}{2\,x\,l}$. For a fixed value of x between these limits, Eq. 43 represents the straight line which makes the intercepts $\frac{P\,h}{M_o} = \frac{l}{x}$ and $\frac{w\,l^2}{8\,M_o} = \frac{l^2}{4\,x\,(l-x)}$ on the coordinate axes in Fig. 26. The envelope EF of all these lines corresponding to various values of x between $l/2$ and l is readily sketched when a few lines corresponding to selected values of x have been drawn. Since all collapse modes that may occur under the considered type of loading are of the kinds shown in Figs. 25(*b*) and 25(*c*), or combinations of them, the preceding discussion is exhaustive. The safe states of loading, therefore, are represented by the points of the shaded area ODEF in Fig. 26.

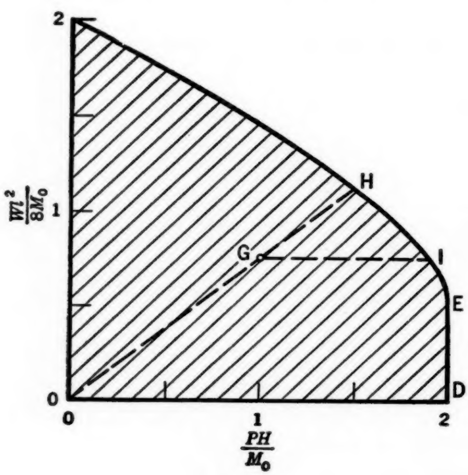

FIG. 26

Let the point G, in Fig. 26, represent the given state of loading . If P and w are to be increased in proportion until collapse occurs the safety factor is the ratio of the abscissas of the points H and G, Fig. 26. On the other hand, if w represents a dead load and only P is to be increased until collapse occurs, the safety factor is the ratio of the abscissas of the points I and G.

The difficulties created by the presence of axial forces have been stressed by several discussers. As Mr. Popov states[39(a)], it is assumed that all com-

[39(a)] Correction for *Transactions*, through the courtesy of Mr. Popov: At the center of the lower horizontal in Fig. 8, change "1" to "11/12".

pression members possess a sufficiently low slenderness ratio so that instability does not occur. To those feeling that this assumption deprives the limit analysis of its value, it must be emphasized that the customary elastic analysis is based on precisely the same assumption. Indeed, the deformations of the members are disregarded when the equilibrium conditions are set up. This means that the possibility of buckling is arbitrarily excluded in the determination of the redundant forces or couples. Only after the axial forces have been found in this manner is the possibility of buckling investigated. The same procedure can be followed in the case of limit analysis. In both cases, the results of the preceding analysis can be accepted only if the subsequent check shows that there is not danger of instability. Otherwise, the cross section of the critical member must be increased, and the first analysis must be repeated regardless of whether it was of elastic or limit type.

Aside from raising the stability question, axial forces are important because they lower the limit moments. Again, the writers were perfectly aware of this fact but did not wish to complicate the exposition of their basic principles by the inclusion of secondary considerations.

When Mr. Franciosi's M_o and N_o are identified with Mr. Benjamin's $1.5\, S\, \sigma_y$ and $A\sigma_y$, respectively, Eq. 32 is seen to represent the parabola in Fig. 15. The straight line in Fig. 15 can be obtained in a similar manner for an I-beam section if the flanges are considered as indefinitely thin and the contribution of the web is neglected. Mr. Franciosi shows how this type of information can be incorporated in the statical analysis based on Theorem I. An alternative kinematic analysis (not published) based on Theorem III has been developed by E. T. Onat at Brown University, Providence, R. I. In this analysis the effect of axial forces are taken into account by replacing the ordinary yield hinges of limit analysis by modified hinges which allow not only a relative rotation but also an approach or separation of adjacent cross sections.

Mr. Hrennikoff's remarks are concerned with one aspect of an important limitation of the theory. Limit analysis is based on the assumption that the beam or frame under consideration can fail only when sufficient yield hinges have developed to transform it into a mechanism. As R. Hill[40] has shown, this assumption is completely justified only in the case of a rigid-plastic beam that cannot bend until the bending monent has reached the value M_o and may then bend indefinitely under the constant bending moment M_o. Of course, this concept of a rigid-plastic beam is not realistic. On the other hand, a beam with the more realistic moment-curvature diagram of Fig. 1 may cease to be serviceable before the collapse load furnished by limit analysis is reached. This may be due to excessive local strain at a yield hinge, as Mr. Hrennikoff states, or to the occurrence of large deflections made possible by the cooperation of several yield hinges, none of which bends excessively. The development of a general, simple method for estimating the deformation at the instant of impending collapse would constitute, therefore, a most significant contribution to the theory of limit analysis.

[40] "On the State of Stress in a Plastic-Rigid Body at the Yield Point," by R. Hill, *Philosophical Magazine*, Series 7, Vol. 42, 1951 p. 868.

For the sake of brevity, the writers assumed the same absolute value M_o for the limit moments for positive and negative curvature. Actually, the methods presented in the paper are readily extended to the case discussed by Mr. Tachau in which the positive limit moment differs in absolute value from the negative limit moment. In the statical analysis based on Theorem I, the sign of the bending moment at each yield hinge must then be taken into account in determining the limit moment for this hinge. In the kinematic analysis based on Theorem III, the sense of the relative rotation at each yield hinge must be considered in determining the limit moment at this hinge and its work on the relative rotation.